

MACROSCOPIC DIMENSION AND FUNDAMENTAL GROUP OF MANIFOLDS WITH POSITIVE ISOTROPIC CURVATURE

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ABSTRACT. We prove a conjecture of Gromov's to the effect that manifolds with isotropic curvature $K_{\mathbb{C}}^{isotr}(M) \geq \epsilon^{-2}$ and a lower bound on Ric are macroscopically 1-dimensional on the scale $\gg \epsilon$. As a consequence we prove that compact manifolds with positive isotropic curvature have virtually free fundamental groups. Our main technique is modeled on Donaldson's version of Hörmander technique to produce (almost) holomorphic sections which we use to construct destabilizing sections.

1. INTRODUCTION

Given a Riemannian manifold (M, g) , one can extend the metric tensor in two ways to the complexified tangent bundle $TM \otimes \mathbb{C}$: as a complex bilinear (\cdot, \cdot) form or as a Hermitian form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. A tangent vector v is called *isotropic* if $(v, v) = 0$, and analogously a 2-plane $\pi \subset TM \otimes \mathbb{C}$ is called *totally isotropic* if $(u, v) = 0$ for every $u, v \in \pi$.

If we view the Riemannian curvature tensor R , as a *quadratic form* Rm on $\bigwedge^2 TM$ (this is the *curvature operator*), we can clearly extend it to a quadratic form $\text{Rm}_{\mathbb{C}}$ on $\bigwedge^2(TM \otimes \mathbb{C})$.

The *sectional curvature* $K(u, v) := \langle R(s, u)u, s \rangle$, inasmuch as the restriction of Rm on bivectors, can then be extended to the complexified tangent bundle. In other words, if we think of the sectional curvature as a function K on $Gr(2, TM)$ —the Grassmannian bundle of 2-planes in TM —we can extend it as a function $K_{\mathbb{C}}$ to the Grassmannian bundle of complex 2-planes in $TM \otimes \mathbb{C}$ $Gr_{\mathbb{C}}(2, TM \otimes \mathbb{C})$, as follows:

$$K_{\mathbb{C}}(\pi) := \text{Rm}(v \wedge w, \overline{v \wedge w})$$

where v and w are two vectors in π which are orthogonal with respect to the Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. Restricting the function $K_{\mathbb{C}}$ to the subbundle of *totally isotropic* two-planes $Gr_{\mathbb{C}}^{isotr}(2, TM)$ (which is non-empty only if $\dim(M) \geq 4$) we obtain the *isotropic curvature* $K_{\mathbb{C}}^{isotr}$.

We are now ready for

Definition 1. We say that M has *positive isotropic curvature* if $K_{\mathbb{C}}^{isotr} > 0$ and that the isotropic curvature is bounded below by k if $K_{\mathbb{C}}^{isotr} \geq k$

These conditions are readily seen to be equivalent to the requirement that for any *orthonormal* 4-frame $\{e_1, e_2, e_3, e_4\}$ one has:

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} + 2R_{1234} > 0 \quad \text{resp.} \quad > k$$

where $R_{ijkl} = \text{Rm}(e_i, e_j, e_k, e_l)$.

Positivity of the isotropic curvature $K_{\mathbb{C}}^{isotr}$ is implied by the positivity of the complex sectional curvature $K_{\mathbb{C}}$ (which in turn is implied by the strong condition of positivity of the Riemannian curvature operator), and it implies the positivity of the *scalar curvature*, nonetheless there are examples of manifolds with positive sectional curvature, or even positive *Ricci curvature*, which do not admit positive isotropic curvature, and more importantly there are examples of manifolds which admit positive isotropic curvature, but which cannot admit a metric with positive Ricci curvature. For instance, in dimension $n = 4$, every locally conformally flat manifold with positive scalar curvature has a metric with positive isotropic curvature, as shown in [29]. More explicitly, $M_k := \#_{i=1}^k S^3 \times S^1$ has a metric with positive isotropic curvature for any k (indeed, as proved in [27], the connected sum of any two manifolds with uniformly positive isotropic curvature admits such a metric), but for topological reasons (which are a consequence of the splitting theorem of Cheeger-Gromoll), it cannot support a metric with *non-negative Ricci curvature*. On the other hand $\mathbb{CP}^2 \#_{i=1}^k \overline{\mathbb{CP}}^2$ admits no metric such that $K_{\mathbb{C}}^{isotr} \geq 0$ for $1 \leq k \leq 9$ (cf. [29]), nonetheless it admits a metric with $Ric > 0$ by Yau's proof of Calabi's conjecture (as it easy to show that the *canonical class* is positive); in fact, even a *Kähler-Einstein* one (necessarely with positive scalar curvature) if $k > 2$ (cf. [36]).

The main result in [26] (cf. [2] for a nice survey) is the following strong topological restriction imposed by positive isotropic curvature:

Theorem 2. (Micallef-Moore) *A compact manifold M with positive isotropic curvature is such that $\pi_i(M) = (0)$ for $2 \leq i \leq [\frac{n}{2}]$, where $[x]$ indicates the integral part of x .*

The connected sum of manifolds with $K_{\mathbb{C}}^{isotr} \geq k > 0$ also admits such a metric (cf. [27]). $S^1 \times S^n$ and spherical space forms admit such metrics. The main conjecture for manifolds with positive isotropic curvature is that, so far as the fundamental group is concerned, these connected sums are the only manifolds with positive isotropic curvature. More precisely, the following conjecture has been put forth:

Conjecture. *The fundamental group of a compact manifold with positive isotropic curvature is residually free.*

A particular case of this conjecture, to the effect that the fundamental group of such manifolds cannot contain subgroups abstractly isomorphic to the fundamental group of a Riemann surface of genus $p \geq 1$, has been proved by A. Fraser–J. Wolfson (cf. [14]) (the special case $p = 1$ had been previously proved by A. Fraser in the foundational paper [13]).

Following Gromov (cf. [20]) we define:

Definition. A metric space V has *macroscopic dimension at most k on the scales $\gg \epsilon$* , if there exists a k -dimensional polyhedron P and a continuous map $\phi : V \rightarrow P$, such that for every fiber $\phi^{-1}(p) \subset V$, $\text{diam } \phi^{-1}(p) \leq \epsilon$. If this is the case, one writes $\dim_\epsilon V \leq k$. Moreover, one sets:

$$\dim_\epsilon V := \sup\{k : \dim_\epsilon V \leq k\}$$

The prospective of Gromov’s in [20] on manifolds with positive curvature through his concept of *macroscopic dimension* turns out to be particularly fruitful in answering the above mentioned conjecture, especially when combined with the ideas of Gromov–Lawson (cf. [21]) and the stability inequality due to Micallef and Moore (cf. [26]).

In fact, we can show the following conjecture of Gromov’s (cf. [20], para. 3):

Theorem 3. *If $K_{\mathbb{C}}^{\text{isotr}}(M) \geq \epsilon^{-2}$, $\text{Ric}(g) \geq -Cg$, $\dim(M) \geq 4$. Then M is macroscopically 1-dimensional on the scale $\gg \epsilon$. In particular, when finitely generated (e.g., M compact) $\pi_1(M)$ is residually free.*

One should remark that the assumption that the isotropic curvature be strictly (and uniformly) positive cannot be relaxed, as shown by the example $M = \Sigma \times S^k$, which admits a metric with *non-negative* isotropic curvature—here Σ is a Riemann surface of genus $p \geq 2$ and S^k is the round k -sphere.

One of the main ingredients—just like in Fraser’s fundamental paper [13]—is the second variation formula of the energy functional, or better yet its manifestation in the form of the *stability inequality* (cf. eq.(2)), as described in [26], which we will explain briefly in section 2.

Another major –and arguably more important for this paper– ingredient comes into play in the construction of the *destabilizing* sections and is based on using Donaldson’s version of Hörmander techniques to construct suitable *almost holomorphic* sections (cf. [10] and [11]). Our construction is closely based on Donaldson’s (cf. also Donaldson–Sun), with the difference that we need to make sure that the section also be *isotropic*. The philosophy is mostly based on Donaldson’s construction

of almost holomorphic sections in [10], the major difference being that unlike in [10], we work with integral complex structures, as in [11], but even in this case we do not need the full blown Hörmander technique as we merely need the sections to be almost holomorphic. We thus effectively first construct highly peaked *holomorphic sections* (i.e., concentrated on a very small ball) following Donaldson's argument but in a way that highly resembles Tian's construction of peak sections (cf. [38]).

Finally we use a cut-off function argument to render the sections thus constructed compactly supported. The estimate we prove using these ingredients – and itself the major ingredient in proving Theorem 3 – is

Theorem 4. *Assume that $K_{\mathbb{C}}^{\text{isotr}}(M) \geq \epsilon^{-2}$, $\text{inj}(g) \geq \iota$ and that $\dim(M) \geq 4$. Let $f : D \rightarrow M$ be a stable, minimal (possibly branched) immersion. Then for every point $p \in D$ there exists a smooth isotropic section $\sigma = \sigma_p$ of E , and a constant C such that:*

$$(1) \quad \frac{\int_D |\nabla_{\frac{\partial}{\partial \bar{z}}} \sigma|^2 dV}{\int_D |\sigma|^2 dV} \leq K \frac{1}{r^2}.$$

where $r := \text{dist}_D(p, D)$. Furthermore, the constant K is computable and it only depends on the geometry of (M, g) .

Corollary 5. *Assume that $K_{\mathbb{C}}^{\text{isotr}}(M) \geq \epsilon^{-2}$, $\text{inj}(g) \geq \iota$ and that $\dim(M) \geq 4$. Then for every closed curve γ such that $[\gamma] = 0$ in $H_1(M)$, then:*

$$\text{FillRad} \gamma \leq C\epsilon$$

An immediate corollary of this coupled with Theorem 1.2 in [31], is the following

Theorem 6. *If M is a closed manifold such that $K_{\mathbb{C}}^{\text{isotr}}(M) \geq \epsilon^{-2}$ then $\pi_1(M)$ is residually free.*

Also, using the main theorem of [15] one has the following immediate consequence:

Theorem 7. *Let M be a closed, orientable Riemannian n -manifold with positive isotropic curvature of dimension $n \geq 5$. Then there exists a finite cover of M which is homeomorphic to the connected sum of k copies of $S^{n-1} \times S^1$.*

Finally, we would like to point out that in dimension 4 something much stronger is true: using the Ricci flow, R. Hamilton (cf. [23])

and B.-L. Chen and X.P. Zhu (cf.[4]) have been able to prove that a compact 4-manifold with positive isotropic curvature and containing no essential incompressible 3-dimensional space form, is *diffeomorphic* to S^4 , $S^3 \times S^1$, \mathbb{RP}^4 and $S^3 \tilde{\times} S^1$ and their connected sums (naturally, the last two do not occur in the oriented case). In fact in a recent beautiful paper, Chen, Tang and Zhu (cf. [5])—using Hamilton’s Ricci flow—have proven the 4-dimensional version of a very far reaching conjecture due to R. Schoen, which claims that the (much stronger) differential version of Theorem 7 should hold.

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2. THE SECOND VARIATION FORMULA

Let $f : \Sigma \rightarrow M$ be a stable minimal surface in M . Consider the pull-back of the tangent bundle with the pull back of the metric and (resp. normal) connection ∇ . Let $E = f^*TM \otimes \mathbb{C}$ be the complexified bundle. The metric on $f^*(T_M)$ extends as a complex bilinear form (\cdot, \cdot) or as a Hermitian metric $\langle \cdot, \cdot \rangle$ on E , and the connection ∇ (the pull-back via f of the Levi-Civita connection of M) and curvature tensor extend complex linearly to sections of E . Moreover the connection is Hermitian with respect to $\langle \cdot, \cdot \rangle$.

By a well known theorem, (cf. [22] and [1]), there is a unique holomorphic structure on E such that the $\bar{\partial}$ operator

$$\bar{\partial} : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E),$$

where $\mathcal{A}^{p,q}(E)$ denotes the space of (p, q) -forms on Σ with values in E , is given by

$$\bar{\partial}\omega = (\nabla_{\frac{\partial}{\partial \bar{z}}} \omega) d\bar{z}$$

where $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, for local coordinates x, y on Σ .

One can choose a metric $ds^2 = \lambda(dx^2 + dy^2)$ on Σ compatible with the conformal structure determined by the pull-back of $G = f^*g$ of the metric g on M via the immersion f . Then one has:

$$\|\nabla_{\frac{\partial}{\partial \bar{z}}} W\|^2 dV_{ds^2} = |\nabla_{\frac{\partial}{\partial \bar{z}}} |_H^2 dx \wedge dy$$

where by $|\nabla_{\frac{\partial}{\partial \bar{z}}}|_H^2$ we mean $\langle \nabla_{\frac{\partial}{\partial \bar{z}}}, \nabla_{\frac{\partial}{\partial \bar{z}}} \rangle$, the Hermitian scalar product induced by f^*g on E . Suppose $f : \Sigma \rightarrow M$ is a *stable* minimal immersion. Then the complexified stability inequality (see [35], [25], [26], [2] and [13]) for the *Energy functional* reads:

$$\int_{\Sigma} \langle R(s, \frac{\partial f}{\partial z}) \frac{\partial f}{\partial \bar{z}}, s \rangle dx \wedge dy \leq \int_{\Sigma} |\nabla_{\frac{\partial}{\partial \bar{z}}} s|^2 dx \wedge dy$$

for all $s \in \mathcal{A}_0^\infty(E)$, the space of smooth sections of E with compact support (if the surface has boundary) or for all $s \in \mathcal{A}^\infty(E)$, the space of smooth sections, if Σ is closed. Assume now that s is *isotropic*. Since f is conformal, $\frac{\partial f}{\partial z}$ is isotropic and $\{s, \frac{\partial f}{\partial z}\}$ spans an isotropic two-plane. If the isotropic curvature is such that $K_{\mathbb{C}}^{isotr} \geq \epsilon^{-2}$, we get:

$$(2) \quad \epsilon^{-2} \int_{\Sigma} |s|^2 dV \leq \int_{\Sigma} |\nabla_{\frac{\partial}{\partial \bar{z}}} s|^2 dV$$

where dV denotes the area element for the induced metric f^*g on Σ . Here the norms are $\frac{1}{\lambda^2}$ times the corresponding norms coming from $TM \otimes \mathbb{C}$ (cf. [26]). We will also denote by $N := \nu_f \otimes \mathbb{C}$ where ν_f is the normal bundle of f , i.e. the bundle defined by the exact sequence of real bundles:

$$0 \rightarrow T_{\Sigma} \rightarrow f^*TM \rightarrow \nu_f \rightarrow 0.$$

The same considerations we did for E hold for N and as observed by A. Fraser in [12], the suability inequality can be formulated as:

$$(3) \quad \epsilon^{-2} \int_{\Sigma} |s|^2 dV \leq \int_{\Sigma} |\nabla_{\frac{\partial}{\partial \bar{z}}}^{\perp} s|^2 dV$$

for any compactly supported section s of N and here ∇^{\perp} is the connection induced to the normal bundle from the Levi-Civita connection.

3. THE TEST SECTIONS

Throughout this section, $f : D \rightarrow M$ will be a stable, minimal (possibly branched) *proper* immersion from the *disk* D to M . We will also maintain the notation of section 2: $E := f^*(TM \otimes \mathbb{C})$ etc. Let q_D be the quadratic form on E induced from the \mathbb{C} -bilinear form (\cdot, \cdot) . If γ is a smooth curve in D , then we denote by q_{γ} the restriction of q_D to $E|_{\gamma}$. Furthermore we will call a smooth section α of $E|_{\gamma}$ *isotropic* if $q_{\gamma}(\alpha, \alpha) = 0$.

3.1. Curvature of Hermitian metrics on Riemann surfaces. Recall that given a Hermitian holomorphic bundle (E, H) one has a unique connection ∇ whose $(0, 1)$ -part $\nabla^{(0,1)}$ is equal to $\bar{\partial}$ (the operator determining the integrable complex structure of E): the Hermitian connection. If $\{e_1, \dots, e_n\}$ is a *holomorphic frame*, and if:

$$h_{i\bar{j}} := H(e_i, \bar{e}_j)$$

then the connection 1-form and the curvature 2-form (which is an $\text{End}(E)$ valued 2-form) are given (respectively) by:

$$(4) \quad A = \partial h \cdot h^{-1}, \quad \Theta_H := \bar{\partial} H = -\partial \bar{\partial} h \cdot h^{-1} + \partial h \cdot h^{-1} \wedge \bar{\partial} h \cdot h^{-1}$$

The curvature tensor— which is a section of $E^* \otimes \bar{E}^* \otimes \Omega_N^{(1,0)} \otimes \Omega_N^{(0,1)}$ where $\Omega_N^{(p,q)}$ is the space of (p, q) -forms— is given by:

$$R(H)(v, w, s, \bar{t}) := H(\Theta_H(s \wedge \bar{t})v, \bar{w})$$

and in the frame $\{e_1, \dots, e_n\}$ and (local) holomorphic coordinates z_1, \dots, z_m on N :

$$(5) \quad R_{i\bar{j}\alpha\bar{\beta}} := R(e_i, \bar{e}_j, \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}) = -\frac{\partial^2 h_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta} + \frac{\partial h_{i\bar{t}}}{\partial z^\alpha} h^{s\bar{t}} \frac{\partial h_{s\bar{j}}}{\partial \bar{z}^\beta}$$

On a Riemann surface, having fixed an holomorphic coordinate z , we simply denote the curvature:

$$(6) \quad R_{i\bar{j}} := R(e_i, \bar{e}_j, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = -\frac{\partial^2 h_{i\bar{j}}}{\partial z \partial \bar{z}} + \frac{\partial h_{i\bar{t}}}{\partial z} h^{s\bar{t}} \frac{\partial h_{s\bar{j}}}{\partial \bar{z}}$$

One can define the *Ricci curvature* of h , denoted $\text{Ric}(h)$, as follows (cf. [39] ch.3 or [24] section 1 for this notion). Let $\{e_\sigma\}$ be a holomorphic frame for E and $\{V_i\}$ any frame field of type $(1, 0)$ relative to some fixed Hermitian metric g on N ; then set:

$$(7) \quad \text{Ric}_h(V_i, V_j) := \sum_{\alpha, \nu} h^{\alpha\bar{\beta}} R(e_\alpha, \bar{e}_\beta, V_i, \bar{V}_j)$$

where $h_{i\bar{j}} := h(e_\nu, e_\xi)$. In components, this is simply:

$$R_{\alpha\bar{\beta}} := \text{Ric}(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}) = h^{i\bar{j}} R_{i\bar{j}\alpha\bar{\beta}}.$$

We also set:

$$(8) \quad K_{i\bar{j}}(g, h) := g^{\alpha\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}.$$

which is referred to as the *mean curvature form* of (E, h) over (N, g) (cf. [24] section 1). We observe that if $N = \Sigma$ is a Riemann surface, then the tensor $K_{i\bar{j}}$ and the full curvature tensor $R_{i\bar{j}\alpha\bar{\beta}}$ for a holomorphic Hermitian vector bundle (E, h) are equivalent. Note that the notion of

Ricci curvature coincides with the standard notion of Ricci curvature, in case E is the tangent bundle of N .

One has readily:

Lemma 8. *Let (M, g) be a Riemannian manifold with $\text{Ric}(g) \geq -C$ (resp. $\text{Rm}(g) \geq -C$). Given a minimal immersion $f : D \rightarrow M$, let $E = f^*TM \otimes \mathbb{C}$ with the induced holomorphic structure and hermitian structure H . Then the Ricci curvature (resp. curvature tensor) of (E, H) :*

$$\text{Ric}(H) \geq -C \quad (\text{resp. } \text{Rm}(h) \geq -C)$$

Moreover, if (M, g) is assumed to have isotropic curvature bounded from below by C_I the for every holomorphic isotropic section σ of E such that $\sigma \wedge \frac{\partial f}{\partial \bar{z}} \neq 0$, one has:

$$(9) \quad R(H)(\sigma, \sigma) := R(H)\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \sigma, \bar{\sigma}\right) \geq -C_I \lambda \|\sigma\|^2$$

where $g = \lambda(dx^2 + dy^2)$ is the (necessarily Kähler) metric on D induced -via f - from (M, g) .

Proof. The only non-trivial components of the curvature of the Hermitian metric H on E are, for any section W of E :

$$R(H)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)W = \left(\nabla_{\frac{\partial}{\partial x}} \nabla_{\frac{\partial}{\partial y}} - \nabla_{\frac{\partial}{\partial y}} \nabla_{\frac{\partial}{\partial x}}\right)W$$

but in terms of the curvature of g these are equal to:

$$R_g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)W$$

evaluated along $f(D)$. Finally, equation (9) is a simple consequence of the computation above and the definition of isotropic curvature on the isotropic plane $\sigma \wedge \frac{\partial f}{\partial \bar{z}} \neq 0$,. \square

Also a very important standard fact is the following (cf. [18] pg. 78-79):

Lemma 9. *Let (E, H) be a holomorphic Hermitian vector bundle and $\pi : E \rightarrow Q$ a holomorphic quotient bundle endowed with the quotient Hermitian metric H_Q . Let also $F \subset E$ an holomorphic sub bundle such that the quotient $E/F \simeq Q$ and H_F the induced Hermitian metric. Then for the curvature operator:*

$$\Theta(H_Q) = \Theta(H) |_Q + S \wedge S^*$$

where $S = \nabla_E - \nabla_F$ is the second fundamental form -here ∇_E and ∇_F are the metric connections of (E, H) and (F, H_F) respectively. In

particular:

$$\Theta(H_Q) \geq \Theta(H) \mid_Q .$$

3.2. Bochner technique in Complex Differential Geometry. When working with a vector bundle E we will often use the norms defined by the rescaled metrics $R^2 g$, which has the effect of rescaling lengths by R and volumes by R^{2n} .

We will use the notation $g_R := R^2 g$ to denote the rescaled metric and we will further simplify notation by writing dV_R for dV_{g_R} , the corresponding volume form. Then the scaling weight gives

$$\|\nabla f\|_{L^2(dV_R)} = R^{n-1} \|\nabla f\|_{L^2(dV_g)} \quad \text{and} \quad \|f\|_{L^2(dV_{g_R})} = R^n \|f\|_{L^2(dV_g)}.$$

We will make use of the following various forms of Laplacian operators:

$$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*,$$

with adjoints defined using the L^2 -metric induced from g

$$\Delta_{R, \bar{\partial}} = \bar{\partial}_R^* \bar{\partial} + \bar{\partial} \bar{\partial}_R^*,$$

with adjoints defined using the rescaled metric g_R . The Laplace-Beltrami operator associated to the connection ∇ is

$$\Delta = \nabla \nabla^* + \nabla^* \nabla$$

where

$$\nabla^* : C^\infty(M, \Lambda^q T_M^* \otimes E) \rightarrow C^\infty(M, \Lambda^{q-1} T_M^* \otimes E)$$

is the (formal) adjoint of ∇ with respect to the L^2 inner product.

If M is a compact complex manifold equipped with a hermitian metric $\omega = \sum \omega_{j\bar{k}} dz_j \wedge d\bar{z}_k$ and E is a holomorphic vector bundle on M equipped with a hermitian metric, and let $\nabla = \nabla' + \nabla''$ be its Chern curvature form (that is to say $\nabla'' = \bar{\partial}$), decomposed in $(1, 0)$ and $(0, 1)$ parts respectively. In the same way one forms the Laplace-Beltrami operator Δ , one can form the following complex Laplace operators:

$$\Delta' = \nabla' \nabla'^* + \nabla'^* \nabla', \quad \Delta'' = \nabla'' \nabla''^* + \nabla''^* \nabla''.$$

The main identity we will be using is:

Theorem 10 (Bochner-Kodaira-Nakano identity). *If (X, ω) is Kähler, the complex Laplace operators Δ' and Δ'' acting on E -valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\sqrt{-1} \Theta(h), \Lambda].$$

One more important piece of information is the following (cf. [9], Ch. VII, eq. (7.1)):

Lemma 11. *If E is a Hermitian vector bundle on a Riemann surface Σ , then $[\sqrt{-1}\Theta(h), \Lambda] = K(g, h)$ where $K(g, h)$ is defined in equation (8).*

3.3. Rescaling. As before, we are given $(f : D \rightarrow M, E, H)$ a triple consisting of a stable minimal immersion $f : D \rightarrow M$, the induced vector bundle $E := f^*TM \otimes \mathbb{C}$ with induced complex structure and with induced Hermitian metric H , and connection A_E . Also, D is endowed with the pull-back metric $g_D = \lambda^2(dx^2 + dy^2)$ (necessarily Kähler by virtue of dimension).

Let $L = D \times \mathbb{C}$ endowed with the Hermitian metric $|\rho|^2 := e^{-\frac{|z|^2}{2}} \rho \bar{\rho}$ compatible with the connection:

$$A := \frac{1}{2} \sum_{i=1}^n (z_i d\bar{z}_i - \bar{z}_i dz_i).$$

We will actually endow $E \otimes L^k = E$ with the tensor product metric $H_k := e^{-\frac{k|z|^2}{2}} H$ and the tensor product connection:

$$A_E \otimes A_k = A_E \otimes \frac{k}{2} \sum_{i=1}^n (z_i d\bar{z}_i - \bar{z}_i dz_i)$$

We will be performing two kinds of rescalings (intimately connected to one another). We either rescale the background metric $g_D = \lambda^2(dx^2 + dy^2)$ by considering the (holomorphic) homotetic transformation, for any point $z_0 \in D$:

$$\Phi_{R,z_0} : D_R \rightarrow D, \text{ defined by } \Phi_{R,z_0}(z) := z_0 + \frac{z}{r}$$

and pulling back all the geometric quantities:

$$g_{D_R} := \Phi_{R,z_0}^* g_D; \quad E_R := \Phi_{R,z_0}^* E; \quad H_R := \Phi_{R,z_0}^* \left(e^{-\frac{|z|^2}{2}} H \right);$$

$$A_R = \Phi_{R,z_0}^* (A_E \otimes A).$$

Otherwise we can consider the rescaling induced by considering the geometric quantities $A_E \otimes A_k$ and H_k defined above. Clearly, by taking $k = \frac{1}{R}$ when H is the standard Euclidean Hermitian metric on $E = D \times \mathbb{C}^n$, these two notions coincide.

3.4. Tweaking the Hermitian metric so it has positive curvature. For most of our constructions it will be convenient to be able to conformally change –by a uniformly controlled conformal factor– the Hermitian metric on E so that the curvature becomes positive. This is achieved by:

Proposition 12. *Let ω be a Kähler form on the disc $D \subset \mathbb{C}$ and let (E, H) be a Hermitian rank n vector bundle over D (hence necessarily trivial), and assume that its curvature satisfies:*

$$R(H) \geq -\theta\omega, \quad (\text{resp.}) \quad \text{Ric}(H) \geq -\theta\omega$$

for some positive number C . Then there exists a conformal Hermitian metric $H_\psi := e^{-\psi}H$ on E such that:

$$R_{H_\psi} \geq 2\omega \quad (\text{resp.}) \quad \text{Ric}(H) \geq 2\omega$$

that is to say the bundle $\det(E)$ endowed with metric $h_\psi := \det(e^{-\psi} H)$ has positive curvature. Furthermore, ψ can be chosen so that:

$$\|\psi\|_{C^{k,\alpha}} < C(\theta)$$

for a constant $C = C(\theta)$ depending only on θ and ω . In particular the oscillation of ψ is uniformly bounded.

Proof. This is based on two facts: on the one hand the elementary linear algebra fact that if A is an $n \times n$ symmetric matrix, then there exist a $k \in \mathbb{R}_+$ such that $A + k I_n > 0$ (where I_n is the identity $n \times n$ matrix); on the other hand the fact, discussed

Since the curvature of $H_\psi := e^{-\psi}H$ is calculated—making use of equation (5) and the fact that $-\partial\bar{\partial}(e^{-\psi}) = (\partial\bar{\partial}\psi - \partial\psi \wedge \bar{\partial}\psi)e^{-\psi}$ —as follows (given that E has rank n):

$$\Theta(H_\psi) = (\Theta(H) + \partial\bar{\partial}\psi H) e^{-\psi}$$

or in coordinates:

$$R_{i\bar{j}}(H_\psi) = \left(R_{i\bar{j}} + \frac{\partial^2 \psi}{\partial z \partial \bar{z}} H_{i\bar{j}} \right) e^{-\psi}$$

after tracing with respect to H , for the Hermitian metric H_ψ to satisfy the conclusion of the theorem it is (necessary and) sufficient that:

$$\Delta_{\text{flat}}\psi = \frac{4}{n}k$$

where k is a function (which without loss of generality we may assume to have a sign) such that $-\theta\omega + k\omega \geq 2\omega$ and $\Delta_{\text{flat}}\psi = 4\partial\bar{\partial}\psi$ is the Laplacian with respect to the flat metric $ds^2 = dx^2 + dy^2$. On the other hand, by standard elliptic theory, given any smooth k and any boundary value ρ , we can find a smooth solution to:

$$\begin{cases} \Delta_{\text{flat}}\psi = \frac{4}{n}k \\ \psi|_{\partial B} = \rho \end{cases}$$

As for the assertion on the oscillation, observe that by (Calderon-Zygmund) elliptic regularity:

$$\|\psi\|_{W^{\ell+2,p}(B)} \leq C(B, \ell) \left(\|\psi\|_{L^2} + \left\| \frac{k}{n} \right\|_{W^{\ell+2,p}(B)} + \|\rho\|_{W^{\ell+2,p}(B)} \right)$$

We remark that since a disk of radius R , $D_R \subset \mathbb{C}$ is strictly pseudoconvex, one can actually choose a plurisub-harmonic defining function χ (e.g. $\chi = |z|^2$) such that $\chi|_{\partial D_R} = R$ and then by taking a suitable multiple $\chi = e^{-C|z|^2}$, one can choose $\rho = CR$ (and in fact $\psi = C|z|^2$) in the above construction. \square

Remark 13. Our application of this Lemma will be to minimal immersions of the disc into our manifold M . Note that for any (possibly branched) minimal immersion $f : D \rightarrow M$ the vector bundle E with the induced (Hermitian) metric from M —which we indicate by $H := f^*g$ —has Ricci curvature depending only on the Ricci curvature of the metric g of M ; hence $\text{Ric}(H)$ is bounded below by some constant C_1

We can also prove the following (which is of independent interest):

Lemma 14. *Keeping notation and assumptions as in Proposition 12, if one also has:*

$$|\text{Ric}_H| < C$$

then there exists a rank 1 holomorphic line sub-bundle $L \subset E$ such that, if h_L denotes the induced metric its curvature satisfies: $\sup_{D'} R(h_L) > -C_1$ for any compactly embedded $D' \subset D$, where C_1 only depends on C and the Sobolev constant and D' .

Proof. In order to prove this we notice that there must exist a holomorphic frame $\{e_1, \dots, e_n\}$ such that:

$$R(h)_{1\bar{1}} := R(h)(e_1, \bar{e}_1, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) > -C$$

Since the Ricci curvature of h equals:

$$\text{Ric}(h) = -h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial z \partial \bar{z}} + h^{i\bar{j}} h^{s\bar{t}} \frac{\partial h_{s\bar{j}}}{\partial z} \frac{\partial h_{i\bar{t}}}{\partial \bar{z}}$$

—which is clearly an elliptic system— it follows from the Calderon-Zygmund inequality that (cf. [30]):

$$\|h_{i\bar{j}}\|_{W^{2,p}(D')} \leq C_Z(D') \|\text{Ric}(h)\|_p$$

and the conclusion follows from Morrey's version of Sobolev inequality, which implies that:

$$\|h^{s\bar{t}} \frac{\partial h_{s\bar{j}}}{\partial z} \frac{\partial h_{i\bar{t}}}{\partial \bar{z}}\|_{C^{0,\alpha}(D')} \leq C_S C_Z(D') \|\text{Ric}(h)\|_p$$

where C_S is the Sobolev constant of the (compact) embedding $W^{2,p} \subset C^{1,\alpha}$ when $\frac{1-\alpha}{2} = \frac{1}{p}$. \square

3.5. Constructing holomorphic isotropic sections. In this section we fix the standard flat metric with Kähler form $\Omega_0 = \sqrt{-1} dz \wedge d\bar{z}$ on $D = D_R \subset \mathbb{C}$, the disk of radius 1.

We analyze the datum of a holomorphic vector bundle $E = D \times \mathbb{C}^n \rightarrow D_R$ over D_R and endowed with a Hermitian metric $\frac{\kappa}{2}H_0 \leq H \leq \kappa H_0$ such that and $\|\nabla H\|_{H_0}, \|\nabla^2 H\|_{H_0} < \kappa$ where the norms are measured with respect to the standard flat metric on the trivial rank n complex vector bundle $D \times \mathbb{C}^n$ —i.e., $H_0(\xi, \xi) := \sum_{i,j} (H_0)_{i\bar{j}} \xi_i \bar{\xi}_j = \sum_{i,j} \delta_{ij} \xi_i \bar{\xi}_j$, where δ_{ij} is the identity matrix—This is the situation that one might achieve by rescaling a small ball centered and contained inside the unit ball.

We intend to show that we can always find *holomorphic* sections of E —although clearly they will not have compact support, whence they will not be test sections for the stability inequality— which are also $g_{\mathbb{C}}$ -isotropic (here $g_{\mathbb{C}}$ is the \mathbb{C} -linear extension of the metric corresponding to the Hermitian metric H) and that we can do so with controlled L^∞ and $W^{1,\infty}$ Sobolev-norms. More specifically: Let:

$$\mathcal{OI} := \left\{ s \in C^\infty(E) : \nabla_{\frac{\partial}{\partial \bar{z}}} s = 0 \text{ and } g_{\mathbb{C}}(s, s) = 0 \right\}$$

and:

$$\mathcal{I}_{S^1} := \{ s \in C^\infty(E|_{S^1}) : g_{\mathbb{C}}(s, s) = 0 \}$$

Proposition 15. *Let H and $g_{\mathbb{C}}$ be as above. There exists a surjective map:*

$$T : \mathcal{OI} \rightarrow \mathcal{I}_{S^1}$$

Furthermore we can find a set of boundary data such that the corresponding counter-images via T are holomorphic isotropic sections s of E such that:

- (1) $|s(0)| = 1$
- (2) $|s|_H^2 \leq 2$
- (3) $|\partial s|_H^2 \leq 2$
- (4) $|s(z)| \geq \frac{1}{2}$ if $z \in B_{\frac{1}{2}}(0)$
- (5) ∂s is $g_{\mathbb{C}}$ -isotropic.

Proof. The map T is simply given by restricting a given section s to the boundary:

$$T(s) := s|_\gamma$$

where $\gamma := \partial D \simeq S^1$. In order to show surjectivity, we first show we can solve the $\bar{\partial}$ -problem for any boundary condition:

$$(11) \quad \begin{cases} \frac{\partial}{\partial \bar{z}} s = 0 \text{ in } D \\ s|_{\gamma} = \chi \end{cases}$$

where $\chi \in C^\infty(\Delta)$ is to be specified later (here $\gamma := \partial D$). That we can solve this equation with any boundary condition is guaranteed by the fact that E is holomorphically trivial on D (cf. Theorem Y pg. 211 in [17]), and therefore the problem reduces to the 1-dimensional Cauchy-Riemann problem for functions on D , which can be solved using the Cauchy integral formula:

$$s_i(\zeta) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{\chi_i(z)}{z - \zeta} dz$$

where –after identifying E with $D \times \mathbb{C}^n$ – s_i are the components of s and $\chi = (\chi_1, \dots, \chi_n)$. We next show that if $\chi \in \mathcal{I}_{S^1}$ then the solution of (11) is in \mathcal{OI} . In order to make sure that s be isotropic only in terms of the boundary condition we make use of the fact that if s is a holomorphic section of E , then:

$$\bar{\partial} g_{\mathbb{C}}(s, s) = 0$$

i.e., $g_{\mathbb{C}}(s, s)$ is holomorphic, which descends immediately from the fact that:

$$\bar{\partial} g_{\mathbb{C}}(s, s) = g_{\mathbb{C}}(\nabla_{\frac{\partial}{\partial \bar{z}}} s, s) + g_{\mathbb{C}}(s, \nabla_{\frac{\partial}{\partial \bar{z}}} s) = 0$$

Indeed, we can choose $\chi = (\chi_1, \dots, \chi_n)$ such that $g_{\mathbb{C}}|_{\gamma}(\chi, \chi) = 0$, therefore by analytic continuation, also s , the solution to the Cauchy problem $\bar{\partial}s = 0$ and $s|_{\partial D} = \chi$, is such that $g_{\mathbb{C}}(s, s) = 0$; i.e., s is an *isotropic* section. This shows the surjectivity of the map T .

We now show that we can choose the boundary data χ so that (1)–(4) hold.

Part (1) and (2) now follow from choosing the χ_i 's accordingly as follows. In the global trivialization chosen:

$$(12) \quad \begin{aligned} |s|_H^2(z) &= H_{i\bar{j}}(z) s^i \bar{s}^j \\ &= H_{i\bar{j}}(z) \left(\frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{\chi_i(z)}{z - \zeta} dz \right) \overline{\left(\frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{\chi_j(z)}{z - \zeta} dz \right)} \end{aligned}$$

and since in polar coordinates centered at $0 \in D$ (i.e., $z = re^{i\theta}$) there holds $\frac{dz}{z}|_{\gamma} = \sqrt{-1}d\theta$, in order for (1) to hold it suffices to require:

$$(13) \quad \sum_{i,j=1}^n H_{i\bar{j}}(0) \left(\frac{1}{2\pi} \int_0^{2\pi} \chi_i(\theta) d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} \bar{\chi}_j(\theta) d\theta \right) = 1$$

We show the existence of such χ as follows. We choose a smooth frame e_i for E on D such that at 0:

$$H(e_i, e_j)(0) = H_{i\bar{j}}(0) = \delta_{ij}$$

In this frame we choose a smooth section of $E|_{\partial D}$ of the form $\tilde{\chi} = (\tilde{\chi}_1, \dots, \tilde{\chi}_n)$ and we have chosen $\tilde{\chi}_i$ such that:

$$g_{\mathbb{C}}(\tilde{\chi}, \tilde{\chi}) = g_{\mathbb{C}ij} \tilde{\chi}_i \tilde{\chi}_j = 0.$$

This is tantamount to choosing $\tilde{\chi} = \tilde{\alpha} + \sqrt{-1}\tilde{\beta}$ with $\tilde{\alpha}$ and $\tilde{\beta}$ sections of $\partial D \times \mathbb{R}^n$ such that:

$$\|\tilde{\alpha}\|_g^2 = \|\tilde{\beta}\|_g^2 \text{ and } \langle \tilde{\alpha}, \tilde{\beta} \rangle_g = 0$$

where g is the real form of the Hermitian metric H .

Set $\chi_i := e^{\sqrt{-1}\lambda\theta} \tilde{\chi}_i$, $\alpha := e^{\sqrt{-1}\lambda\theta} \tilde{\alpha}$, $\beta := e^{\sqrt{-1}\lambda\theta} \tilde{\beta}$ and $\chi := (\chi_i, \dots, \chi_n)$. Clearly one still has that:

$$\|\alpha\|_g^2 = \|\tilde{\alpha}\|_g^2 = \|\tilde{\beta}\|_g^2 = \|\beta\|_g^2 \text{ and } \langle \alpha, \beta \rangle_g = 0$$

or equivalently:

$$g_{\mathbb{C}}(\chi, \chi) = 0 \text{ and } \|\chi\|_H^2 = \|\tilde{\chi}\|_H^2.$$

Since the span of α and β in $\partial D \times \mathbb{C}^n$ is a rank 2 vector bundle $F = \langle \alpha \rangle \oplus \langle \beta \rangle$ isomorphic to $\partial D \times \mathbb{R}^2$, the metric on this sub-bundle is conformal to the standard flat metric. That is to say we can find a complex Gauge such that $H|_F = e^\phi h_0$ where $h_0(\eta, \xi) = \eta_1 \bar{\xi}_2 + \eta_2 \bar{\xi}_1$.

In particular, since by assumption on H we must have $\frac{1}{2} \leq e^\phi \leq \frac{3}{2}$, one can readily show that one can have chosen $\tilde{\chi} = \tilde{\alpha} + \sqrt{-1}\tilde{\beta}$ satisfying the following¹:

- the following inequality about the mean of the components:

$$(14) \quad 2 \leq \sum_{i,j=1}^n H_{i\bar{j}}(0) \left(\frac{1}{2\pi} \int_0^{2\pi} \tilde{\chi}_i(\theta) d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} \bar{\tilde{\chi}}_j(\theta) d\theta \right) \leq 4$$

- the necessary conditions for $\tilde{\chi}$ to be *isotropic*:

$$(15) \quad \|\tilde{\alpha}\|_g^2 = \|\tilde{\beta}\|_g^2 = \frac{1}{2} \text{ and } \langle \tilde{\alpha}, \tilde{\beta} \rangle_g = 0$$

- and finally that:

$$(16) \quad \sup_{\partial D} |\tilde{\chi}|_H^2 \leq \frac{3}{2}$$

¹For instance the choices $\tilde{\alpha} := (\tilde{\alpha}_1, \tilde{\alpha}_2, 0, \dots, 0) = \frac{1}{2}(-1, 1, 0, \dots, 0)e^{-\frac{\phi}{2}}$ and $\tilde{\beta} := (\tilde{\beta}_1, \tilde{\beta}_2, 0, \dots, 0) = \frac{1}{2}(1, 1, 0, \dots, 0)e^{-\frac{\phi}{2}}$ and hence $\tilde{\chi} = \frac{1}{2}(-1 + \sqrt{-1}, 1 + \sqrt{-1}, 0, \dots, 0)e^{-\frac{\phi}{2}}$ will do the trick.

Since:

$$I_\lambda := \sum_{i,j=1}^n H_{i\bar{j}}(0) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}\lambda\theta} \tilde{\chi}_i(\theta) d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-\sqrt{-1}\lambda\theta} \overline{\tilde{\chi}_j(\theta)} d\theta \right)$$

is a continuous expression in λ and since $\lim_{\lambda \rightarrow \infty} I_\lambda = 0$ (viewing $\tilde{\chi}_i$ as periodic functions on \mathbb{R} with period 2π and for $\lambda \in \mathbb{N}$ these are simply the Fourier coefficients) it follows that there exists a choice of λ for which:

$$\sum_{i,j=1}^n H_{i\bar{j}}(0) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}\lambda\theta} \chi_i(\theta) d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-\sqrt{-1}\lambda\theta} \bar{\chi}_j(\theta) d\theta \right) = 1$$

That is to say equation (13) holds. Then if s is the solution of the Dirichlet problem with $\chi := (\chi_i, \dots, \chi_n)$ as boundary condition (i.e., $s = T^{-1}(\chi)$) one can easily check that there is a choice of λ such that:

$$\begin{aligned} |s|_H^2(0) &= H_{i\bar{j}}(0) s^i \bar{s}^j = |T^{-1}(\chi)|_H^2(0) \\ &= \sum_{i,j=1}^n H_{i\bar{j}}(0) \left(\frac{1}{2\pi} \int_0^{2\pi} \chi_i(\theta) d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} \bar{\chi}_j(\theta) d\theta \right) = 1 \end{aligned}$$

We next prove that (2) holds in two different ways. In the first proof we simply exploit the assumption that $\frac{\kappa}{2} H_0 \leq H \leq \kappa H_0$. In the second (which we only sketch) one employs that the curvature of H is bounded (as consequence of the fact that by assumption $\frac{\kappa}{2} H_0 \leq H \leq \kappa H_0$ and $\|\nabla H\|_{H_0}, \|\nabla^2 H\|_{H_0} < \kappa$). One easily proves (e.g., choosing a frame at any given $p \in D$ where $H_{i\bar{j}}(p) = \delta_{ij}$, $\partial H(p) = \bar{\partial} H(p) = 0$ and $\frac{\partial^2 H_{i\bar{j}}}{\partial z \partial \bar{z}}(p) = -R(H)_{i\bar{j}}$) the following Bochner type formula:

$$(17) \quad \partial \bar{\partial} |s|_H^2 = -R(H)_{i\bar{j}} s^i \bar{s}^j + |\partial s|_H^2$$

We can now proceed by observing that on the one hand:

$$(18) \quad |s|_H^2 \leq \kappa |s|_{H_0}^2$$

and that on the other hand $-R(H_0)_{i\bar{j}} = 0$, therefore, using equation (17) and applying the maximum principle to $|s|_{H_0}^2$, yields:

$$\sup_D |s|_{H_0}^2 = \sup_{\partial D} |s|_{H_0}^2$$

whence (coupled with eq. (18)):

$$|s|_H^2 \leq \kappa |s|_{H_0}^2 \leq \kappa \sup_{\partial D} |s|_{H_0}^2 \leq 2 \sup_{\partial D} |s|_H^2$$

having used that $\frac{\kappa}{2} H_0 \leq H$.

Since by equation (15) :

$$(19) \quad \|\alpha\|_g^2 = \|\beta\|_g^2 = \frac{1}{2}$$

thus, using eq. (12):

$$\begin{aligned} & |s|_H^2(\zeta) \\ & \leq 2 \sup_{\zeta \in \partial D} \left(H_{i\bar{j}}(\zeta) \left(\frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{\chi_i(z)}{z - \zeta} dz \right) \overline{\left(\frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \frac{\chi_j(z)}{z - \zeta} dz \right)} \right) \\ & \leq 2 \sup_{\partial D} (\|\alpha\|_H^2 + \|\beta\|_H^2) = 2 \end{aligned}$$

Item (3) is a consequence of the inequality (itself a consequence of the Cauchy-integral formula):

$$\left| \frac{\partial^k}{\partial z^k} s_i \right|(x_0) \leq \frac{k!}{R^k} \sup_{\partial B(0,R)} |\chi|$$

for any ball $B(x_0, R)$. In particular, making use of inequality (16):

$$|\partial s|_H^2 \leq \kappa |\partial s|_{H_0}^2 \leq \kappa \sup_{\partial D} \|\chi\|_{H_0}^2 \leq 2 \sup_{\partial D} \|\chi\|_H^2$$

where in the next to last inequality we employ the fact that by choice (see footnote on page 13) $\chi = e^{-\phi} v$ for some constant vector v .

Finally, having bounded the gradient of s , and therefore bounded the Lipschitz constant of $|s|$, item (4) follows.

The second proof (which we only sketch) is based on an argument similar to the one employed in the proof of Proposition 12, which we can use to show that we can conformally change H so that $R(e^{-\phi} H)_{i\bar{j}} < 0$ which holds for any holomorphic section s .

Finally we prove item (5). It is not difficult to show that the boundary data can be chosen as required, but we give a different (and more direct) proof of existence here. Next consider any *holomorphic* sub-bundle F of E of complex rank equal to 1 generated by an *isotropic holomorphic section*. Since the map T in the first part of the proposition is surjective, there are plenty such holomorphic complex line sub-bundles (as many as \mathcal{I}_{S^1}).

To fix the notation, we assume that we have chosen a unitary frame $\{e_1, \dots, e_n\}$ for $E \simeq D \times \mathbb{C}^n$ such that F is given by $e_2 = e_3 = \dots = e_n = 0$ (i.e., F is generated by the holomorphic isotropic section α and $e_1 := \frac{\alpha}{\|\alpha\|}$): hence e_1 is isotropic and via the (non-holomorphic) vector bundle isomorphism $\Phi : E \rightarrow D \times \mathbb{R}^{2n}$ induced by the unitary frame $\{e_1, \dots, e_n\}$, e_1 corresponds to $(1, 0, \dots, 0)$

On F the metric H takes the form $e^{-\phi}h_0$ where h_0 is the standard flat metric: $h_0(\xi, \xi) = \xi \bar{\xi}$. Let $A = -\partial\phi + \bar{\partial}\phi$ be the connection 1-form. Then clearly the complex structure (which is conjugate to the standard one) is determined (in our non-holomorphic frame) by the $\bar{\partial}$ -operator:

$$\bar{\partial}_A := \bar{\partial} + \bar{\partial}\phi$$

and in this frame:

$$\partial_A = \partial - \partial\phi$$

so that $d_A = d + A$. Now on $F = \Delta \times \mathbb{R}^2$ we have that $s := e^{-\phi}(\tilde{s}_1, 0, \dots, 0)$ with $\bar{\partial}\tilde{s}_1 = 0$ is $\bar{\partial}_A$ -holomorphic and:

$$\begin{aligned} \partial_A(e^{-\phi}(\tilde{s}_1, 0, 0, \dots, 0)) &= -2\partial\phi e^{-\phi}(\tilde{s}_1, 0, \dots, 0) \\ &\quad + e^{-\phi}(\partial\tilde{s}_1, 0, \dots, 0) \end{aligned}$$

thus if \tilde{s}_1 is chosen to be any constant complex numbers, one readily sees that s satisfies items (1)-(5). \square

Remark 16. This proposition is similar in spirit to Lemma 2.1 in [6], except here we make sure we can find solutions with controlled norm (specifically bounded away from zero) at least on a smaller disk. Also here we exploit directly the holomorphic triviality of holomorphic vector bundles on disk (more generally polydisks) rather than solving the Riemann-Hilbert problem for the coupled $\bar{\partial}$ operator.

3.6. The model example: isotropic holomorphic Gaussian sections. This is a very slight modification of Donaldson's technique. We merely want to make sure that the "local" construction produces **isotropic** holomorphic sections.

Let then B_R be the ball of radius $R > 2$ in \mathbb{C}^N (in the application we will take $N = 1$) with the standard flat metric and standard Kähler form: $\Omega_0 = \sqrt{-1} \sum_i dz_i \wedge d\bar{z}_i$. Let F be the trivial rank n holomorphic vector bundle $F = B_R \times \mathbb{C}^n$ with metric conformal to the flat Hermitian metric by the factor $\exp(-|z|^2/2)$. The 1-form:

$$A_k := \frac{k}{2} \sum_{i=1}^n (z_i d\bar{z}_i - \bar{z}_i dz_i)$$

gives rise to, a diagonal connection $A_{n,K} := \oplus_{i=1}^n A_{k_i}$ on F for any multi-index $K = (k_1, \dots, k_n)$.

The curvature F_{A_k} of A_k is simply $dA_k = -\sqrt{-1}k\Omega_0$ and there is (up to constant rescaling) only one Hermitian metric H_k on $B_R \times \mathbb{C}$ compatible with A_k (i.e., the metric H_k whose curvature is F_{A_k}): the metric $H_k = e^{-\frac{k|z|^2}{2}}h_0$ where h_0 is the standard flat Hermitian metric on $B_R \times \mathbb{C}$.

Observe that the connection $A_{n,K}$ gives rise to a $\bar{\partial}_{A_{n,K}}$ operator on $B_R \times \mathbb{C}^n$ as follows:

$$\bar{\partial}_{A_{n,K}}(s_1, \dots, s_n) = (\bar{\partial}_{A_{k_1}} s_1, \dots, \bar{\partial}_{A_{k_n}} s_n)$$

where $\bar{\partial}_{A_{k_i}} s = \bar{\partial} + A_{k_i}^{0,1} s$ (here $A_{k_i}^{0,1}$ indicates the $(0,1)$ -part of A_{k_i}), and also to a Hermitian metric on F whose curvature is $dA_{n,K} = -\sqrt{-1} \bigoplus_{i=1}^n k_i \Omega_0$, namely: $H_{n,K} = \bigoplus_{i=1}^n e^{-\frac{k_i |z|^2}{2}} H_{0,K}$ where $H_{0,K}$ the standard flat Hermitian metric on $F = B_R \times \mathbb{C}$: i.e., the diagonal matrix $H_{0,k} := \text{diag}(k_1, \dots, k_n)$. Since homotheties are holomorphic, from now on we restrict ourselves to considering multi indexes of the form $K = (k, \dots, k) = k(1, \dots, 1)$.

Definition 17. Let:

$$g_{\mathbb{C}}(\sigma, \sigma) := \exp(-k|z|^2/2) \left(\sum \sigma_i \sigma_i \right)$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$. A section of F , σ say, is said to be **isotropic** if:

$$g_{\mathbb{C}}(\sigma, \sigma) = 0.$$

Remark 18. Clearly $g_{\mathbb{C}}$ is the \mathbb{C} -linear extension of the metric on $B_R \times \mathbb{R}^n$ whose Hermitian extension is $H_{n,k}$.

In what follows we will find it convenient to switch between different representations (corresponding to different gauges): one where we might view as having fixed the Hermitian metric H_0 and having represented the complex structure on F as $\bar{\partial}_{A_{n,k}}$, or equivalently we will fix the complex structure (which in our trivializing charts is the standard one) and then consider the Hermitian metric on F as being given by $H_{n,k}$ (the equivalence of these views can be seen as an incarnation of the Poincare- Lelong formula for $\partial\bar{\partial} \log |s|_H^2$ for a Hermitian metric H and a holomorphic section s). We are now ready to prove:

Lemma 19. *On the (necessarily) trivial rank n holomorphic bundle F over B_R , the ball of radius R , for any multi index K there is a connection $A_{n,K}$ (with associated Hermitian metric $H_{n,K}$) and smooth section σ such that:*

- (1) $F_{A_{n,K}} = dA_{n,K} + [A_{n,K}, A_{n,K}] = -\sqrt{-1} \bigoplus_i k_i \Omega$, where $K = (k_1, \dots, k_n)$
- (2) $\bar{\partial}_{A_{n,K}} \sigma = 0$
- (3) σ is isotropic: $g_{\mathbb{C}}(\sigma, \sigma) = 0$.
- (4) $|\sigma(0)|_{H_{n,K}} = 1$
- (5) $|\sigma|_{H_{n,K}}^2 = e^{-\frac{|z|^2}{2}} |\sigma|_{H_{0,K}}$

- (6) $|\sigma(z)|_{H_{n,k}} \geq \frac{1}{2}$ if $z \in B_{\frac{1}{2}}(0)$.
 (7) $\partial\sigma$ is isotropic: $g_{\mathbb{C}}(\partial\sigma, \partial\sigma) = 0$.

Proof. Set:

$$A_n := \oplus_{i=1}^n A$$

as above. Then clearly (since $[A_n, A_n] = 0$):

$$F_{A_n} = dA_n = \oplus dA = -\sqrt{-1}\omega^{\oplus n}.$$

Write:

$$\sigma := \exp(-|z|^2/2)(\sigma_{0,1}, \dots, \sigma_{0,n})$$

where $\sigma_0 = (\sigma_{0,1}, \dots, \sigma_{0,n})$ is a *holomorphic isotropic* section such that (as constructed in Proposition 15):

- $|\sigma_0(0)| = 1$
- $|\sigma_0|_g^2 \leq 1$
- $|\partial\sigma_0|_g^2 \leq 1$
- $|\sigma_0(z)| \geq \frac{1}{3}$ if $z \in B_{\frac{1}{2}}(0)$.

Then σ is A_K -holomorphic (i.e., item (2) holds):

$$(20) \quad \begin{aligned} \bar{\partial}_{A_K}(\exp(-|z|^2/2)\sigma_{0,i}) &= \\ &= \bar{\partial}(\exp(-|z|^2/2))\sigma_{0,i} + A^{(0,1)}\exp(-|z|^2/2)\sigma_{0,i} = 0 \end{aligned}$$

for every $i \in \{1, \dots, n\}$, having used that $\bar{\partial}\sigma_{0,i} = 0$. The norm of σ is $\exp(-|z|^2/4)|\sigma_0|^2$ and σ is isotropic with respect to $g_{\mathbb{C}}$ (i.e., item (3) holds). Also:

$$\begin{aligned} \|\partial_{A_K}(\exp(-|z|^2/2)\sigma_{0,i})\| &= \\ &= \|\partial(\exp(-|z|^2/2))\sigma_{0,i} + \exp(-|z|^2/2)\partial(\sigma_{0,i}) + A^{(1,0)}\exp(-|z|^2/2)\sigma_{0,i}\| \\ &\leq \|\partial(\exp(-|z|^2/2))\| \|\sigma_{0,i}\| + \|\exp(-|z|^2/2)\| \|\partial(\sigma_{0,i})\| \\ &\quad + \|A^{(1,0)}\| \|\exp(-|z|^2/2)\sigma_{0,i}\| \end{aligned}$$

The rest is a straightforward application of Proposition 15. \square

Remark 20. It is important to remark that the positivity of the curvature is what produces the Gaussian type holomorphic sections. For instance, on the line bundle $L = D \times \mathbb{C}$ with Hermitain metric $e^{|z|^2}h_0$ with negative curvature, then the holomorphic sections one produces have exponential growth.

We can now prove:

Proposition 21. *In the same assumptions as Lemma 19, there exists an $R > 0$, a smooth section, s of F such that:*

- (1) $\pi < \|s\|_{L^2} < 2\pi$ and $|\|s\|_{L^2} - \|s\|_{L^2(G)}| \leq \frac{1}{10}$ where $G = dx^2 + dy^2$.
- (2) $|s(0)| = 1$;
- (3) For any smooth section τ of F over a neighborhood of \overline{D} we have

$$|\tau(0)| \leq C (\|\bar{\partial}\tau\|_{L^p(D)} + \|\tau\|_{L^2(D)});$$

$$(4) \quad \|\bar{\partial}s\|_{L^2} < \frac{3}{R} \left(\|s\|_{L^2(B_R)} + e^{-\frac{R^2}{2}} 2\pi R \right);$$

$$(5) \quad \partial s \text{ is isotropic on } B_{\frac{R}{2}}, \text{ the ball of radius } \frac{R}{2}$$

Proof. First of all, item (3) holds for a C independent of R . In fact, given $D' \subset D$ some interior domain containing 0, the standard elliptic estimate

$$(21) \quad \|\tau\|_{L_1^p(D_0)} \leq C_e (\|\bar{\partial}\tau\|_{L^p(D)} + \|\tau\|_{L^2(D')}) ,$$

coupled with the Sobolev inequality

$$|\tau(u_*)| \leq C_S \|\tau\|_{L_1^p(D')}.$$

yield (3).

Let $\eta_R = \eta_R(|z|)$ be a cut-off function:

$$\eta_R(|z|) = \begin{cases} 1 & \text{if } |z| \leq R/2 \\ 0 & \text{if } |z| \geq \frac{9R}{10} \end{cases}.$$

such that:

$$|\eta'_R| \leq \frac{3}{R}$$

Let σ be as in Lemma 19. Define:

$$s = \eta_R \sigma.$$

Then we have $\bar{\partial}s = (\bar{\partial}\eta_R)\sigma$, therefore:

$$(22) \quad \|\bar{\partial}s\| = \|\bar{\partial}\eta_R \sigma\| = |\bar{\partial}\eta_R| \|\sigma\|$$

whence:

$$\begin{aligned} \|\bar{\partial}s\|_{L^2(B_R)} &\leq \frac{3}{R} \|\sigma\|_{L^2(B_R)} = \frac{3}{R} \left(\|\sigma\|_{L^2(B_{\frac{R}{2}})} + \|\sigma\|_{L^2(B_R \setminus B_{\frac{R}{2}})} \right) \\ &\leq \frac{3}{R} \left(\|\sigma\|_{L^2(B_{\frac{R}{2}})} + e^{-\frac{R^2}{2}} \|\sigma_0\|_{L^2(B_R)} \right) \leq \frac{2}{R} \left(\|\sigma\|_{L^2(B_{\frac{R}{2}})} + e^{-\frac{R^2}{2}} 2\pi R \right) \\ &\leq \frac{3}{R} \left(\|s\|_{L^2(B_R)} + e^{-\frac{R^2}{2}} 2\pi R \right) \end{aligned}$$

which proves item (4). One easily verifies that:

$$\|s\|_{L^2} = 2\pi - \delta \quad \|s\|_{L^2(G)} = 2\pi - \delta'$$

for some small, $\delta, \delta' > 0$ – which proves item (1)– and also, trivially:

$$|s(0)| = 1.$$

This proves item (2).

By item (7) in Lemma 19, σ is such that $\partial\sigma$ is isotropic on $B_{\frac{R}{2}}$ as it equals $\partial\sigma$ there. \square

3.7. General facts about holomorphic section of vector bundles. Next we prove a general very well known theorem (cf. [11], where it is stated for line bundles) on holomorphic sections of a Hermitian holomorphic vector bundle on a Kähler manifold (N, g) with the extra assumption that the background metric g satisfies a lower bound on the Ricci curvature. This will not apply immediately to our context, but it will apply to the context in which we endow the disk with the flat metric (or a rescaled version of the flat metric). We will then be able to effect the control desired merely because the sections we construct have L^2 -norm in the background metric which is comparable (by a given and definite amount) to the L^2 -norm with respect to the flat metric.

Proposition 22. *Let (E, H) be a holomorphic Hermitian vector bundle on a Kähler manifold (N, g) and σ a holomorphic section of E such that following conditions hold:*

$$K(H)(\partial\sigma, \partial\sigma) \geq -C_K \quad \text{and} \quad \text{Ric}(g) \geq -C_R$$

where by $K(H)$ we denote the mean curvature of H (cf. section 3.1) Then:

- (1) $\|\sigma\|_{L^\infty(H)} \leq \kappa_0 \|\sigma\|_{L^2(H)}$, $\|\nabla\sigma\|_{L^\infty(H)} \leq \kappa_1 \|\sigma\|_{L^2(H)}$ for some uniform constants κ_0 and κ_1 ;
- (2) $|\sigma(x)| \geq 1/4$ at all points x a distance (in the rescaled metric $g_R := R^2 g$ on N) less than $\min\{\frac{R}{2}, (4\kappa_1)^{-1}\}$ from 0, for some uniform κ_1 depending only on C_K and C_R ;

Proof. We produce a uniform derivative estimate first and then use Moser iteration. This produces a uniform estimates since the lower bound on $\text{Ric}(g)$ entails a uniform control on the Sobolev constant. Let ∇^* and $\bar{\partial}^*$ be calculated with respect to g . First remark that:

$$\nabla^* \nabla s = 2\bar{\partial}^* \bar{\partial} s + s,$$

therefore in the ball $B_{\frac{R}{2}}$ (here the ball is calculated with respect to g_R) –where s is holomorphic– $\nabla^* \nabla s = s$ which implies that (in the sense of barriers):

$$(23) \quad \Delta|s| \leq |s|,$$

since on the one hand:

$$\Delta|s|^2 = 2\langle \nabla^* \nabla s, s \rangle = 2|s|^2$$

and on the other:

$$\Delta|s|^2 = 2|s|\Delta|s| + 2|\nabla|s||^2 \geq 2|s|\Delta|s|.$$

Now the bound on the L^∞ norm follows from the Moser iteration argument applied to this differential inequality (see [37]). Remark that the Sobolev constant here is uniform because of the lower bound on $\text{Ric}(G_R) = 0$, so the bound obtained from Moser iteration is uniform.

Next we derive the first derivative bound, i.e., the second part in item (3), which in turn implies item (4). Again we restrict ourselves to the ball $B_{\frac{R}{2}}$, where s is a holomorphic section, thus $\bar{\partial}s = 0$ and therefore $\nabla s = \partial s$ where

$$\partial : \Omega^{p,q}(E) \rightarrow \Omega^{p+1,q}(E)$$

is defined using the connection. Since $\partial^2 = 0$ we have

$$\Delta_\partial \partial s = \partial \Delta_\partial s,$$

where $\Delta_\partial = \partial^* \partial + \partial \partial^*$. Then for a holomorphic section s , $\Delta_\partial s = \nabla^* \nabla s = s$ and

$$\Delta_\partial(\partial s) = \partial s.$$

The Bochner-Kodaira-Nakano formula (cf. Theorem 10) involving Δ_∂ and $\nabla^* \nabla$ on $\Omega^{1,0}(E)$ is:

$$\Delta_\partial = \nabla^* \nabla - 1 + K(H_\psi)$$

This yields (using that $R(H_\psi) \geq \omega_G$):

$$\nabla^* \nabla = \Delta_\partial + 1 - K(H_\psi) \leq \Delta_\partial + 1$$

so:

$$\begin{aligned} \langle \nabla^* \nabla(\partial s), \partial s \rangle &= \langle \Delta_\partial(\partial s), \partial s \rangle + \langle \partial s, \partial s \rangle - R(H_\psi)(\partial s, \partial s) \\ &= 2|\partial s|^2 - K(H_\psi)(\partial s, \partial s) \leq (C_K + 2)|\partial s|^2. \end{aligned}$$

It follows that

$$\Delta|\partial s| \leq (C_K + 2)|\partial s|,$$

and the Moser argument applies as before. Notice that the constants only depend on the lower bound C_R of the Ricci curvature, the Sobolev constant of the flat metric on the disk and the dimension. Therefore we have shown item (3). Item (4) follows from it, since item (3) bounds the Lipschitz constant of $|s|$. □

With a stronger hypothesis on the structure of the induced metric on the minimal immersion, we can also prove the following:

Theorem 23. *Let $f : \Sigma \rightarrow M$ be a minimal immersion of a compact (not necessarily closed) Riemann surface and assume that the induced metric g_Σ on Σ satisfies:*

$$\text{Ric}(g_\Sigma) \geq -C_\Sigma$$

*If the isotropic curvature of (M, g) satisfies $K_{\mathbb{C}}^{\text{isotr}} \geq -C_K$, then there exists a holomorphic g -isotropic section σ of $E = f^*TM \otimes \mathbb{C} \rightarrow \Sigma$ such that:*

- (1) $1 \leq \|\sigma\|_{L^2} \leq \frac{11}{5}\pi$;
- (2) $\bar{\partial}\sigma = 0$ on the ball of radius $\frac{R}{2}$ (in the rescaled metric);
- (3) $\|\sigma\|_{L^\infty(H)} \leq \kappa_0 \|\sigma\|_{L^2(H)}$, $\|\nabla\sigma\|_{L^\infty(H)} \leq \kappa_1 \|\sigma\|_{L^2(H)}$ for some uniform constants κ_0 and κ_1 ;
- (4) $|\sigma(x)| \geq 1/4$ at all points x a distance (in the rescaled metric g_R) less than $\min\{\frac{R}{2}, (4\kappa_1)^{-1}\}$ from 0, for some uniform κ_1 depending only on C_Σ

Proof. According to Lemma 24 we can find a rescaling $\Psi_k : B_k \rightarrow B$ so that (for k sufficiently large) we can get $E_k \rightarrow B_k$ to satisfy the hypotheses of Theorem 28.

Let $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$, with adjoints defined using the rescaled metric $g_R := \Phi_R^* g = \Phi_R^* (\lambda(dx^2 + dy^2))$, then for all ϕ

$$(24) \quad \langle \Delta_{\bar{\partial}}^{-1} \phi, \phi \rangle_{g_R} \leq \frac{R}{R-C} \|\phi\|_{L^2(g_R)}^2$$

In fact by Kodaira-Nakano:

$$(25) \quad \Delta_{\bar{\partial}} = (\nabla^{(0,1)})^* \nabla^{(0,1)} + K(H, g_{\Sigma, R}) + 1$$

where $g_{\Sigma, R}$ is the result of rescaling g_Σ and $K(H, g_{\Sigma, R})_{\alpha\bar{\beta}} = g_R^{i\bar{j}} R(H)_{i\bar{j}\alpha\bar{\beta}}$. Whence, when restricted to the space of *isotropic* sections:

$$\Delta_{\bar{\partial}} \geq 1 - \frac{C_K}{R}$$

in the operator sense (since $\text{Ric}(g_R) \geq -\frac{C_K}{R}$)

Thus, if we set $s = \sigma - \tau$ where $\tau = \bar{\partial}^* \Delta_{\bar{\partial}}^{-1} \bar{\partial} \sigma$ then clearly $\bar{\partial}s = 0$. Also:

$$\|\tau\|_{L^{2,R}} = \langle \Delta_{\bar{\partial}}^{-1} \bar{\partial} \sigma, \bar{\partial} \bar{\partial}^* \Delta_{\bar{\partial}}^{-1} \bar{\partial} \sigma \rangle = \langle \Delta_{\bar{\partial}}^{-1} \bar{\partial} \sigma, \bar{\partial} \sigma \rangle,$$

since $\bar{\partial} \bar{\partial} \sigma = 0$. Thus

$$(26) \quad \|\tau\|_{L^{2,R}} \leq \sqrt{\frac{R}{R-C_K}} \|\bar{\partial} \sigma\|_{L^{2,R}}$$

Hence in particular, for R sufficiently big:

$$\|s\|_{L^{2,R}} \leq \|\sigma\|_{L^{2,R}} + \|\tau\|_{L^{2,R}} \leq \frac{11}{10} 2\pi$$

Also, since on the ball of radius $\frac{R}{2}$, σ is holomorphic, it follows that $s = \sigma$ on $B_{\frac{R}{2}}$ and therefore it is isotropic there. Since s is holomorphic everywhere so is $g_{\mathbb{C}}(s, s)$ but since s is isotropic on $B_{\frac{R}{2}}$, which is equivalent to $g_{\mathbb{C}}(s, s) = 0$, it follows by analytic continuation that s is isotropic everywhere.

The rest is Proposition 22. \square

3.8. Making the complex structure and the bundle almost standard. Let $B \subset \mathbb{C}$ be the unit ball and let $E \rightarrow D$ a holomorphic vector bundle endowed with a Hermitian metric H . We assume—as we may in the application where $E := f^*TM \times \mathbb{C}$ thanks to Proposition 12—that the curvature of the induced metric $h := \det(H)$ on $\det(E)$ has curvature $\text{Ric}(H) \geq \omega$ for a fixed Kähler form ω_D in D (in the application of course this will be the induced metric from M).

Since $E \rightarrow D$ is a holomorphic line bundle we can infer the fact that E is holomorphically trivial on D (cf. [17]), that is to say there is an isomorphism:

$$\phi : E \rightarrow D \times \mathbb{C}^n$$

Clearly the map ϕ above is not an isometry of bundles. Nonetheless in order to apply Donaldson's philosophy we merely need to construct an *almost isometry*. More precisely, given any point $w_0 \in D$, our approach consists in considering small enough discs centered at w_0 , $B_r := B_r(w_0) \subset D$, such that when rescaling the metric on B_r and on $E|_{B_r}$ by a suitable factor k , the metrics are approximately flat.

To set this up formally, we introduce the following notation. Denote by:

$$\Psi_R : B_R \rightarrow B$$

the standard dilation by R :

$$\Psi_R(z) = w_0 + \frac{z}{R}$$

and let $E_R := \Psi_R^*E$, $g_R := \Psi_R^*g = R^2 \lambda(w_0 + \frac{z}{R})(dx^2 + dy^2)$ (where $g = \lambda(dx^2 + dy^2)$) and $H_R := \Psi_R^*H$ and also $\phi_R := \Psi_R^*\phi$.

Observe that for the curvature form of H_R one has:

$$\begin{aligned} \Theta(H_R)_{i\bar{j}} &= R(H_R)_{i\bar{j}}(z) dz \wedge d\bar{z} = \frac{1}{R^2} R(H)_{i\bar{j}} \left(w_0 + \frac{z}{R} \right) dz \wedge d\bar{z} \\ &= R(H)_{i\bar{j}} \left(w_0 + \frac{z}{R} \right) dw \wedge d\bar{w} \end{aligned}$$

where $R(H)_{i\bar{j}} := R(H)(e_i, \bar{e}_j, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$ are the components of the curvature of H and $\Theta(H)$ is the curvature 2-form.

Let $\Lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$ an n -tuple of positive numbers and consider the endomorphism of \mathbb{C}^n :

$$\Lambda Id_{\mathbb{C}^n} : \mathbb{C}^n \rightarrow \mathbb{C}^n \quad \Lambda Id_{\mathbb{C}^n}(z_1, \dots, z_n) = (\lambda_1 z_1, \dots, \lambda_n z_n)$$

Observe that, for any $\epsilon > 0$, there exists an R sufficiently big, such that:

$$(27) \quad \|K - \Lambda Id_{\mathbb{C}^n}\|_{C^\infty} < \epsilon$$

where Ω_0 denotes the (Kähler form associated to the) flat metric on \mathbb{C}^n . The following fact is now obvious:

Lemma 24. *Up to a constant endomorphism of E , the Hermitian bundle (E, H) is nearly isometric, via Φ_R , to the bundle $B_R \times \mathbb{C}^n$ endowed with the Hermitian metric H (defined up to a constant endomorphism of $D \times \mathbb{C}^n$) whose curvature form is:*

$$\Omega_\Lambda := \oplus_i \lambda_i \Omega_0$$

Proof. First observe that by equation (27) we may assume that in some scale $R_{i\bar{j}}(H)$ and $\Lambda Id_{\mathbb{C}^n}$ are ϵ -close. Since we are on a Riemann surface, the curvature of the metric H takes the form (cf. formula (6))

$$R_{i\bar{j}} := R(e_i, \bar{e}_j, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = -\frac{\partial^2 H_{i\bar{j}}}{\partial z \partial \bar{z}} + \frac{\partial H_{i\bar{t}}}{\partial z} H^{s\bar{t}} \frac{\partial H_{s\bar{j}}}{\partial \bar{z}}$$

By elliptic regularity, given a function $K_{i\bar{j}}$ and a function $h_{i\bar{j}} \in C^{k,\alpha}(\partial D)$, there exists a metric $H_{i\bar{j}} \in C^{k+2,\alpha}(D)$ with $K_{i\bar{j}}$ as curvature, since:

$$(28) \quad \begin{cases} -\frac{\partial^2 H_{i\bar{j}}}{\partial z \partial \bar{z}} + \frac{\partial H_{i\bar{t}}}{\partial z} H^{s\bar{t}} \frac{\partial H_{s\bar{j}}}{\partial \bar{z}} = K_{i\bar{j}} \\ H_{i\bar{j}}|_{\partial D} = h_{i\bar{j}} \end{cases}$$

is an elliptic equation. Furthermore, by the maximum principle, two such solutions differ by a constant endomorphism of $D \times \mathbb{C}^n$. Therefore, up to composing with said endomorphism:

$$\|H - \Omega_\Lambda\| < \epsilon.$$

□

We can now prove:

Proposition 25. *Let (E, H) be a holomorphic Hermitian bundle over $D \subset \mathbb{C}$ (the unit disk) with associated connection 1-form A . Assume that $E = F \otimes \mathbb{C}$ for some real bundle F ² and assume further that F is endowed with an inner product structure g and that its complex bilinear extension $g_{\mathbb{C}}$ is such that $H(v, w) := g_{\mathbb{C}}(v, \bar{w})$.*

²This is automatic on the disk D

If the curvature of H is positive on isotropic two planes, then there exists a smooth section σ such that:

- (1) $\bar{\partial}_A \sigma = 0$, i.e. σ is holomorphic;
- (2) σ is isotropic: $g_{\mathbb{C}}(\sigma, \sigma) = 0$.
- (3) $|\sigma(0)|_H = 1$
- (4) $|\sigma(z)|_H \geq \frac{1}{2}$ if $z \in B_{\frac{1}{2}}(0) \subset D$.
- (5) $\partial \sigma$ is isotropic: $g_{\mathbb{C}}(\partial \sigma, \partial \sigma) = 0$.

Proof. Let $w_0 \in D$ any point in the interior and let $B_r(w_0)$ be a ball centered at w_0 of radius r chosen so that:

$$\sup_{B_r(p)} \|H - CH_0\| < \epsilon$$

for any given $\epsilon > 0$. In fact we achieve this by requiring that after rescaling, using the map Φ_R defined at the beginning of this section, on D_R we can achieve:

$$\sup_{D_R} \|\Theta(H_R) - \Theta(H_R)(w_0)\| < \epsilon$$

for any given $\epsilon > 0$.

We now rescale, using the map Φ_R defined at the beginning of this section, so to make $B_r(w_0)$ the unit disk. According to Lemma 15, we can produce holomorphic *isotropic* sections of E , therefore we can choose one such section σ_0 . Therefore the holomorphic sub bundle E_{iso} of E generated by holomorphic isotropic sections is non-empty.

Let L be the holomorphic line bundle generated by σ_0 . On $L \simeq D \times \mathbb{C}$, the Hermitian structure induced by H is of the form: $e^{-\phi} h_0$ where $h_0(v, w) := v\bar{w}$. Let $p \in D$ any point in the interior and let $B_r(p)$ be a ball centered at p of radius r . We have chosen r so small that $\sup_{B_r(p)} |\phi - C|z|^2| < \epsilon$ for a given $\epsilon > 0$. We can next apply the arguments of Lemma 19 to the holomorphic section $e^{-\phi} \sigma_0$ to prove (1)-(3) and (5). We now achieve the uniform estimate in item (4) by applying Proposition 22 to control uniformly the L^∞ -norm of $\|\nabla s\|$ and therefore the Lipschitz constant of $|\sigma|$.

□

Next we need (in the spirit of Property (H) in [11]):

Proposition 26. *In the same assumptions as above, there exists an $R > 0$, a smooth isotropic section, s of E such that:*

- (1) $\pi < \|s\|_{L^2} < 2\pi$;
- (2) $|s(0)| = 1$;

- (3) For any smooth section τ of E over a neighborhood of \overline{D} we have

$$|\tau(0)| \leq C \left(\|\bar{\partial}\tau\|_{L^p(D)} + \|\tau\|_{L^2(D)} \right);$$

where the volume form is the Euclidian one.

- (4) $\|\bar{\partial}s\|_{L^2} < \frac{3}{R} \left(\|s\|_{L^2(B_R)} + e^{-\frac{R^2}{2}} 2\pi R \right);$

- (5) ∂s is isotropic

Proof. We observe that because of Lemma 15, we can produce holomorphic *isotropic* sections σ of E . Because of Lemma 24, by taking R sufficiently large we may assume that (E, H) is nearly isometric, via Φ_R , to the flat bundle $B_R \times \mathbb{C}^n$ endowed with the standard flat metric $\Omega_\Lambda := \oplus_i \lambda_i \Omega_0$. is small and then we repeat the argument of Proposition 21 but with the constant section replaced by the section σ . The rest is clear. Note that $\partial\sigma$ is also isotropic by construction. \square

The next Proposition is due to Donaldson and Song (cf. [11])

Proposition 27. *The properties (1)-(4) in Proposition 21 are open with respect to variations in (g, J, A) (for fixed $(B, D, 0, F)$) and the topology of convergence in C^0 on compact subsets of U .*

We are now ready to prove the main result of this section. First we define the following sets:

$$\mathcal{M}(\epsilon, C) := \{(M, g) : K_{\mathbb{C}}^{isotr}(M) \geq \epsilon^{-2}\}$$

and

$$\mathcal{K} := \left\{ f : D \rightarrow (M, g) : \begin{array}{l} f \text{ is a minimal proper immersion and} \\ (M, g) \in \mathcal{M}(\epsilon, C) \end{array} \right\}$$

where $D \subset \mathbb{R}^2$ is the unit disc.

Theorem 28. *Suppose that $(D_R, D, 0, F := D_R \times \mathbb{C}^n)$ are as above and the datum g_0, J_0, A_0 satisfies properties (1)-(5) in Proposition 21. Then there is some $\epsilon_0 > 0$ such that if (D, f, M, g) is in \mathcal{K} and If we can find $R > 0$, a scaling $\Phi_R : D_R \rightarrow B$ with $\Phi_R(0) = 0$ and a bundle isomorphism $\hat{\Phi}_R : F \rightarrow E := f^*(TM \otimes \mathbb{C})$ such that*

$$\|\hat{\Phi}_R^*(J) - J_0\|_U, \|\Phi_R^*(g) - g_0\|_U, \|\Phi_R^*(A) - A\|_U \leq \epsilon,$$

with $\epsilon \leq \epsilon_0$ then there is a smooth section s of E such that:

- (1) $\pi < \|s\|_{L^2} < 2\pi;$
 (2) $|s(0)| = 1;$
 (3) For any smooth section τ of F over a neighborhood of \overline{D} we have

$$|\tau(0)| \leq C \left(\|\bar{\partial}\tau\|_{L^p(D)} + \|\tau\|_{L^2(D)} \right);$$

- (4) $\|\bar{\partial}s\|_{L^2} < \frac{3}{R} \left(\|s\|_{L^2(B_R)} + e^{-\frac{R^2}{2}} 2\pi R \right)$;
- (5) $|s(x)| \geq 1/4$ at all points x a distance (in the scaled metric) less than K_1 from 0, for a uniform constant K_1 which only depends on the lower bound of $\text{Ric}(g)$ and the dimension

Proof. Items (1)-(4) are a straightforward consequence of Proposition 27 and Proposition 21. Item (5) follows from Proposition 22 since the L^2 -norm of with respect to Φ_R^*g is comparable to the L^2 -norm calculated with respect to the Φ_R^*G . \square

3.9. The destabilizing section and the Main theorem. In this section we fix the flat metric $G := dx^2 + dy^2$ on D and the metric f^*g on E (here we abuse notation in writing f^*g , meaning the Hermitian metric induced by $H := f^*g$ on $E := f^*TM \otimes \mathbb{C}$). We will also denote by Δ the flat Laplacian on D (i.e., the Laplacian on functions associated to G) and by ∇_ψ the covariant derivative on E corresponding to the Hermitian metric H_ψ on E constructed in Proposition 12.

In order to prove the Main theorem we will need:

Proposition 29. *For any point $p \in f(D)$ such that $r := \text{dist}_{f(D)}(p, \partial f(D))$ there exists a compactly supported g -isotropic section $\sigma = \sigma_p$ of $E \rightarrow D$ such that:*

- (1) *The support of σ_p is contained in the ball of radius r centered at p : $B_r(p)$;*
- (2) $\pi < \|\sigma\|_{L^2} < 2\pi$;
- (3) $|\sigma(0)| = 1$;
- (4) $\|\bar{\partial}\sigma\|_{L^2} < \frac{C}{r} \|s\|_{L^2(B_R)}$;
- (5) $|\sigma(x)| \geq 1/4$ at all points x a distance (in the scaled metric) less than K_1 from 0, for a uniform constant K_1 which only depends on the lower bound of $\text{Ric}(g)$ and the dimension

Proof. We first rescale (and translate) so that $B_r(p)$ is the unit ball centered at 0. According to Lemma 24 we can find a rescaling $\Psi_k : B_k \rightarrow B$ so that (for k sufficiently large) we can get $E_k \rightarrow B_k$ to satisfy the hypotheses of Theorem 28. \square

We can now prove our main estimate:

Theorem 30. *Let $f : D \rightarrow M$ be a stable, minimal (possibly branched) immersion. Then for every point $p \in D$ there exists a smooth isotropic section $\sigma = \sigma_p$ of E which is perpendicular to $\frac{\partial f}{\partial z}$, and a constant C*

such that:

$$(29) \quad \frac{\int_D |\nabla_{\frac{\partial}{\partial \bar{z}}} \sigma|^2 dV}{\int_D |\sigma|^2 dV} \leq C \frac{1}{r^2}.$$

where $r := \text{dist}_D(p, \partial D)$. Furthermore, the constant C is computable and it only depends on the geometry of (M, g) .

Proof. Let us fix notation first. We will view f^*g simultaneously as inducing a Hermitian metric on the vector bundle E and as inducing a (necessarily) Kähler metric on D . In its incarnation as the latter, we will write it as $g = \lambda G = \lambda (dx^2 + dy^2)$. As before, we denote the Hermitian metric induced on E by H .

We will in fact consider the vector bundle $N := \nu_f \otimes \mathbb{C}$ (here ν_f is the normal bundle of the map f , cf. section 2) with the complex structure compatible with ∇^\perp (whose existence is guaranteed by Koszul-Malgrange theorem). Let H_n be the hermitian metric induced by the quotient map $E \rightarrow N$.

Since the curvature of g is *positive* on totally isotropic 2-planes, we infer that for any isotropic section s of N (so that s and $\frac{\partial}{\partial \bar{z}}$ are independent) we have that:

$$K(H)\left(\frac{\partial}{\partial z}, s, \frac{\partial}{\partial \bar{z}}, s\right) > 0$$

therefor a straightforward application of Lemma 9 yields that for any *isotropic* section s of N :

$$K(H_n)\left(\frac{\partial}{\partial z}, s, \frac{\partial}{\partial \bar{z}}, s\right) > 0$$

Given a point $p \in D$ at distance r from the boundary, as in the hypothesis, we consider the ball $B_r(p)$ centered at p of radius r . We then rescale the induced metric $g = \lambda (dx^2 + dy^2)$ on D so that $B_r(p)$ becomes the unit disk D centered at the origin. It now suffices to prove that the inequality (29) in the theorem holds with $r = 1$.

Whence, according to Proposition 29, we can find a smooth $g_{\mathbb{C}}$ -isotropic section σ of $E \rightarrow B$ such that:

- (1) The support of σ_p is contained in the ball of radius r centered at p : $B_r(p)$;
- (2) $\pi < \|\sigma\|_{L^2} < 2\pi$;
- (3) $|\sigma(0)| = 1$;
- (4) $\|\bar{\partial}\sigma\|_{L^2} < \frac{C}{r} \|s\|_{L^2(B_R)}$;

- (5) $|\sigma(x)| \geq 1/4$ at all points x a distance (in the scaled metric) less than K_1 from 0, for a uniform constant K_1 which only depends on the lower bound of $\text{Ric}(g)$ and the dimension

Note that item (2) and (4) imply that:

$$\frac{\int_D |\nabla_{\frac{\partial}{\partial \bar{z}}}^\perp \sigma|_H^2 dV_g}{\int_D |\sigma|_H^2 dV_g} \leq \frac{C}{r^2}$$

□

Finally we remark that with Theorem 4– which we just proved– in hand, the proof of Theorem 3 is a mere application of the techniques of Gromov-Lawson in [21], and in particular of the implication that Theorem 10.2 therein implies Theorem 10.7.

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